Separation of variables

In some cases, you can solve a differential equation

$$f(x, y, y') = 0$$

by moving all the x's to one side and the y's to the other. Then solve the equation by integrating both sides. This is called **separation of variables**.

Example. $x^2\,dx + y(x-1)\,dy = 0$.

Separate:

$$x^2\,dx + y(x-1)\,dy = 0$$

$$-\int \frac{x^2}{x-1}\,dx = \int y\,dy$$

Integrate:

$$-\int \left(x + 1 + \frac{1}{x-1}\right)\,dx = \int y\,dy$$

$$-\left(\frac{1}{2}x^2 + x + \ln|x-1|\right) + C = \frac{1}{2}y^2$$

$$-x^2 - x - 2\ln|x-1| + C_0 = y^2$$

Observe that there is one *integration step*, hence only one constant.

Note also that in the last line I replaced $2C$ with C_0. It would not be wrong to write $2C$, but this is neater. You can always rename constant quantities to make the result look nicer.

Finally, the problem did not include an initial condition; hence, I've stopped at y^2, rather than taking square roots. Without an initial condition, I can't tell which square root to take. $\square$

Example. **(Exponential growth or decay)** Let a be a constant. The **exponential growth or decay equation** describes a situation in which a variable grows or shrinks at a rate proportional to the amount present:

$$\frac{dy}{dx} = ay.$$

Separate:

$$\frac{dy}{dx} = ay, \quad \int \frac{dy}{y} = \int a\,dx.$$

Integrate and solve for y:

$$\ln|y| = ax + C, \quad |y| = e^{ax+C} = e^C e^{ax}, \quad y = C_0 e^{ax}.$$

(I've replaced $\pm e^C$ with C_0.) If $a > 0$, then y increases as x increases: exponential *growth*. If $a < 0$, then y decreases as x decreases: exponential *decay*. $\square$

Example. **(Logistic growth)** In the real world, things cannot grow without bound. In many cases, there is a natural limit to the ability of an environment to support the growth of a population. For example, there are always limits to the food supply and space.

In many cases, this situation is modelled by the **logistic equation**. Let a be a constant. The logistic equation is

$$\frac{dN}{dt} = aN\left(1 - \frac{N}{K}\right).$$

Separate:

$$\frac{dN}{dt} = aN\left(1 - \frac{N}{K}\right)$$

$$\int \frac{dN}{N\left(1 - \dfrac{N}{K}\right)} = \int a\,dt$$

$$K\int \frac{dN}{N(K-N)} = \int a\,dt$$

Compute the integral on the left by partial fractions:

$$\frac{1}{N(K-N)} = \frac{A}{N} + \frac{B}{K-N}$$

$$1 = A(K-N) + BN$$

Set $N = 0$; then $1 = KA$, so $A = \dfrac{1}{K}$. Set $N = K$; $1 = KB$, so $B = \dfrac{1}{K}$. Therefore,

$$\frac{1}{N(K-N)} = \frac{1}{K}\left(\frac{1}{N} + \frac{1}{K-N}\right).$$

Back to the integration:

$$\int \left(\frac{1}{N} + \frac{1}{K-N}\right) dN = \int a\,dt$$

$$\ln|N| - \ln|K-N| = at + C$$

Now solve for N in terms of t:

$$\ln\left|\frac{N}{K-N}\right| = at + C$$

$$\left|\frac{N}{K-N}\right| = e^{at+C} = e^C e^{at}$$

$$\frac{N}{K-N} = C_0 e^{at}$$

$$N = KC_0e^{at} - C_0e^{at}N$$

$$N\left(1 + C_0e^{at}\right) = KC_0e^{at}$$

$$N = \frac{KC_0e^{at}}{1 + C_0e^{at}}$$

Note that $\lim\limits_{t\to\infty} N = K$. Thus, K is the limiting population. It is often called the **carrying capacity**, the largest population that the environment can support. □

Example. (Dropping solutions) Consider the equation

$$\frac{dy}{dx} = (x-3)(y+1)^{2/3}.$$

Separate:

$$\frac{dy}{dx} = (x-3)(y+1)^{2/3}$$
$$\int \frac{dy}{(y+1)^{2/3}} = \int (x-3)\,dx$$

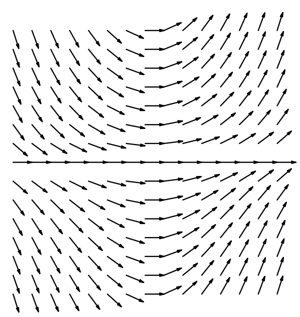
Integrate and solve for y:

$$3(y+1)^{1/3} = \frac{1}{2}(x-3)^2 + C$$
$$(y+1)^{1/3} = \frac{1}{6}(x-3)^2 + C_0$$
$$y+1 = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3$$
$$y = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3 - 1$$

All of this looks routine. However, note that $y = -1$ is a solution to the *original* equation:

$$\frac{dy}{dx} = 0 \quad \text{and} \quad (x-3)(y+1)^{2/3} = 0.$$

You can see the solution $y = -1$ as a horizontal line in the direction field below:



However, you can't obtain $y = -1$ from $y = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3 - 1$ by setting the constant C_0 equal to a number. (You'd need to find a constant which makes $\frac{1}{6}(x-3)^2 + C_0 = 0$ *for all* x.)

Two points emerge from this.

1. You can often drop solutions by performing certain algebraic operations (in this case, division).

2. You don't always get every solution to a differential equation by assigning values to the arbitrary constants. $\square$

Example. **(Equations of the form $y' = f(ax + by + c)$)** A standard rule of thumb is to substitute for an expression which appears "a lot" in an equation or expression. A differential equation

$$y' = f(ax + by + c)$$

can be reduced to a separable equation by the substitution $v = ax + by + c$.

Consider the equation $y' = (x+y+1)^2$. Let $v = x+y+1$, so $v' = 1 + y'$. Then

$$\begin{aligned} v' - 1 &= v^2 \\ \frac{dv}{dx} &= v^2 + 1 \\ \int \frac{dv}{v^2+1} &= \int dx \\ \arctan v &= x + C \\ v &= \tan(x+C) \\ x + y + 1 &= \tan(x+C) \\ y &= \tan(x+C) - x - 1. \quad \square \end{aligned}$$